A LONESOME BOY (from Steve Strogatz, “Infinite Powers”, pp. 186 – 194)

Isaac Newton was born in a stone farmhouse on Christmas Day 1642. Apart from the date, there was nothing auspicious about his arrival. He was born premature and was so tiny, it was said, he could fit inside a quart mug. He was also fatherless. The elder Isaac Newton, a yeoman farmer, had died three months earlier, leaving behind barley, furniture, and some sheep.

When little Isaac was three, his mother, Hannah, remarried and left him in the care of his maternal grandparents. (His mother’s new husband, Reverend Barnabas Smith, insisted on this arrangement; he was a wealthy man twice her age and wanted a young wife but not a young son.) Understandably, Isaac resented his stepfather and felt abandoned by his mother. Later in life, on a list of sins he’d committed before the age of nineteen, he included this entry: “13. Threatning my father and mother Smith to burne them and the house over them.” The next entry was darker: “14. Wishing death and hoping it to some.” And then this: “15. Striking many. 16. Having uncleane thoughts words and actions and dreamese.”

He was a troubled, lonely little boy with no companions and too much time on his hands. He pursued scholarly investigations on his own, building sundials in the farmhouse, measuring the play of light and shadows on the wall. When he was ten, his mother returned, widowed again, with three new children in tow, two daughters and a son. She sent Isaac away to a school in Grantham, eight miles up the road, too far for him to walk each day. He boarded with Mr. William Clark, an apothecary and chemist, from whom he learned cures and remedies, boiling and mixing, and how to grind with a mortar A Lonesome Boy Isaac Newton was born in a stone farmhouse on Christmas Day 1642. Apart from the date, there was nothing auspicious about his arrival. He was born premature and was so tiny, it was said, he could fit inside a quart mug. He was also fatherless. The elder Isaac Newton, a yeoman farmer, had died three months earlier, leaving behind barley, furniture, and some sheep. When little Isaac was three, his mother, Hannah, remarried and left him in the care of his maternal grandparents. (His mother’s new husband, Reverend Barnabas Smith, insisted on this arrangement; he was a wealthy man twice her age and wanted a young wife but not a young son.) Understandably, Isaac resented his stepfather and felt abandoned by his mother. Later in life, on a list of sins he’d committed before the age of nineteen, he included this entry: “13. Threatning my father and mother Smith to burne them and the house over them.” The next entry was darker: “14. Wishing death and hoping it to some.” And then this: “15. Striking many. 16. Having uncleane thoughts words and actions and dreamese.” He was a troubled, lonely little boy with no companions and too much time on his hands. He pursued scholarly investigations on his own, building sundials in the farmhouse, measuring the play of light and shadows on the wall. When he was ten, his mother returned, widowed again, with three new children in tow, two daughters and a son. She sent Isaac away to a school in Grantham, eight miles up the road, too far for him to walk each day. He boarded with Mr. William Clark, an apothecary and chemist, from whom he learned cures and remedies, boiling and mixing, and how to grind with a mortar and pestle.

In 1664 he was awarded a scholarship, and he delved into mathematics in earnest. Teaching himself from six standard texts of the era, he got up to speed on the basics of decimal arithmetic, symbolic algebra, Pythagorean triples, permutations, cubic equations, conic sections, and infinitesimals. Two authors especially enthralled him: Descartes, on analytic geometry and tangents, and John Wallis, on infinity and quadrature.

At Play with Power Series

While poring over Wallis's Arithmetica Infinitorum in the winter of 1664–65, Newton chanced upon something magical. It was a new way to find the areas under curves, a way that was both easy and systematic.

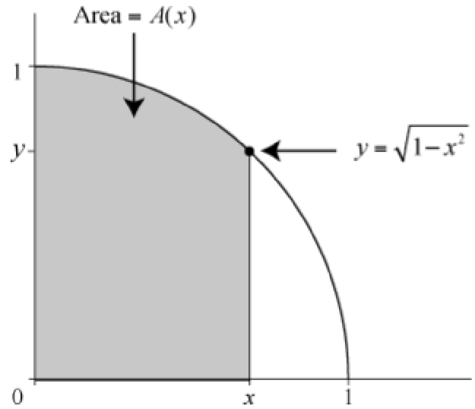
In essence, he turned the Infinity Principle into an algorithm. The traditional Infinity Principle says that to compute a complicated area, reimagine it as an infinite series of simpler areas. Newton followed that strategy, but he updated it by using symbols, not shapes, as his building blocks. Instead of the usual shards or strips or polygons, he used powers of a symbol x, like x^2 and x^3. Today we call his strategy *the method of power series*.

Newton viewed power series as a natural generalization of infinite decimals. An infinite decimal, after all, is nothing but an infinite series of powers of 10 and 1/10. The digits in the number tell us how much of each power of 10 or 1/10 to mix in. For example, the number pi = 3.14 . . . corresponds to this particular mix:

$$3.14\ldots = 3\times10^0 + 1\times\left(\tfrac{1}{10}\right)^1 + 4\times\left(\tfrac{1}{10}\right)^2 + \cdots.$$

Of course, to write any number in this manner, we need to allow ourselves to use infinitely many digits, which is what infinite decimals demand and require. By analogy, Newton suspected he could concoct any curve or function out of infinitely many powers of x. The trick was to figure out how much of each power to mix in. In the course of his studies he developed several methods for finding the right mix.

He hit on his method while thinking about the area of a circle. By making this ancient problem more general, he uncovered a structure within it that nobody had ever noticed before. Rather than restricting his attention to a standard shape, like a whole circle or a quarter circle, he looked at the area of an oddly shaped "circular segment" of width x, where x could be any number from 0 to 1 and where 1 was the radius of the circle.



This was his first creative move. The advantage of using the variable x was that it let Newton adjust the shape of the region continuously, as if by turning a knob. A small value of x near 0 would produce a thin, upright segment of the circle, like a thin strip standing on its edge. Increasing x would fatten the segment into a blocky region. Going all the way up to an x-value of 1 would give him the familiar shape of a quarter circle. By dialing x up or down, he could go anywhere he liked in between.

Through a freewheeling process of experimentation, pattern recognition, and inspired guesswork (a style of thinking he learned from Wallis's book), Newton discovered that the area of the circular segment could be expressed by the following power series:

$$A(x) = x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9 - \cdots.$$

As for where those peculiar fractions came from or why the powers of x were all odd numbers, well, that was Newton's secret sauce. He cooked it up by an argument that can be summarized as follows. (Feel free to skip the rest of this paragraph if you are not especially interested in his argument. However, if you would like to see the details, check out the notes for references.) Newton began his work on the circular segment by using analytic geometry. He expressed the circle as $x^2 + y^2 = 1$ and then solved for y to get

$$y = \sqrt{1 - x^2}.$$

Next he argued that the square root was equivalent to a half power and hence that $y = (1 - x^2)^{1/2}$; note the ½ power to the right of the parenthesis. Then, since neither he nor anyone else knew how to find the areas of segments for half powers, he sidestepped the problem — his second creative move — and solved it for whole powers instead. Finding the areas for *whole* powers was easy; he knew how from his reading of Wallis. So Newton

cranked out the areas of segments for $y = (1 - x^2)^1$ and $(1 - x^2)^2$ and $(1 - x^2)^3$ and so on, all of which have whole-number powers like 1, 2, and 3 outside their parentheses. He expanded the expressions with the binomial theorem and saw that they became sums of simple power functions, the individual area functions of which he had already tabulated, as we saw on the

page from his handwritten notebook. Then he looked for patterns in the areas of the segments as functions of x. Based on what he saw for whole powers, he guessed the answer — his third creative move — for half powers and then checked it in various ways. The answer for the ½ power led him to his formula for A(x), the amazing power series with the peculiar fractions displayed above.

The derivative of the power series for the circular segment then led him to an equally amazing series for the circle itself:

$$y = \sqrt{1 - x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots.$$

There was much more to come, but already this was remarkable. He'd concocted a circle out of infinitely many simpler pieces — simpler, that is, from the standpoint of integration and differentiation. All its ingredients were power functions of the form x^n, where the power n was a whole number. All the individual power functions had easy derivatives and integrals (area functions). Likewise, the numerical values of x^n could be calculated with simple arithmetic using nothing more than repeated multiplication and could then be combined into a series, again using nothing more than addition, subtraction, multiplication, and division. There were no square roots to take or any other messy functions to worry about. If he could find power series like this for *other* curves besides circles, integrating them would become effortless too.

At barely twenty-two, Isaac Newton had found a path to the holy grail. By converting curves to power series, he could find their areas systematically. The backward problem was a cinch for power functions, given the pairs of functions he had tabulated. So any curve that he could express as a series of power

functions was every bit as easy to solve. This was his algorithm. It was tremendously powerful.

Then he tried a different curve, the hyperbola y = 1/(1 + x), and found he could write it, too, as a power series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

This series in turn led him to a power series for the area of a segment under the hyperbola from 0 to x, the hyperbolic counterpart of the circular segment he'd studied earlier. It defined a function that he called the hyperbolic logarithm and that today we call the natural logarithm:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \cdots.$$

Logarithms excited Newton for two reasons. First, they could be used to speed up calculations enormously, and second, they were relevant to a controversial problem in music theory he was working on: how to divide an octave into perfectly equal musical steps without sacrificing the most pleasing harmonies of the traditional scale. (In the jargon of music theory, Newton was using logarithms to assess how faithfully an equal-tempered division of the octave could approximate the traditional tuning of just intonation.)

Thanks to the marvels of the internet and the historians at the Newton Project, you can travel back to 1665 right now and watch young Newton at play.

(His handwritten college notebook is freely available at http://cudl.lib.cam.ac.uk/view/MS-ADD-04000/.) Look over his shoulder at page 223 (105v in the original) and you'll see him comparing musical and geometrical progressions. Zoom in on the bottom of that page to see how he connects his calculations to

logarithms. Then go to page 43 (20r in the original) to watch him "square the hyperbola" and use his power series to calculate the natural logarithm of 1.1 to fifty digits.

What kind of person calculates logarithms by hand to fifty digits? He seemed to be reveling in the newfound strength his power series gave him. When he later reflected on the extravagance of this calculation, he sounded a bit sheepish: "I am ashamed to tell to how many places I carried these computations, having no other business at that time: for then I took really too much delight in these inventions."

If it's any consolation, nobody's perfect. When he first did these computations, Newton made a small arithmetic error. His calculation was correct to only twenty-eight digits. He later caught the error and fixed it.

After his foray with the natural logarithm, Newton extended his power series to the trigonometric functions, which arise whenever circles or cycles or triangles appear, as in astronomy, surveying, and navigation. Here, however, Newton was not the first. More than two centuries earlier, mathematicians in Kerala, India, had discovered power series for the sine, cosine, and arctangent functions. Writing in the early 1500s, Jyesthadeva and Nilakantha Somayaji attributed these formulas to Madhava of Sangamagrama (c. 1350–c. 1425), the founder of the Kerala school of mathematics and astronomy, who

derived them and expressed them in verse about two hundred and fifty years before Newton. In a way it makes sense that power series should have been anticipated in India. Decimals were also developed in India, and as we've seen, Newton regarded what he was doing for curves as an analog of what infinite decimals had done for arithmetic.

The point of all this is that Newton's power series gave him a Swiss army knife for calculus. With them, he could do integrals, find roots of algebraic equations, and calculate the values of non-algebraic functions like sines, cosines, and logarithms. As he put it, "By their help analysis reaches, I might almost say, to all problems."

Newton as Mash-Up Artist

I don't believe Newton was consciously aware of it, but in his work on power series he behaved like a mathematical mash-up artist. He approached area problems in geometry via the Infinity Principle of the ancient Greeks and infused it with Indian decimals, Islamic algebra, and French analytic geometry.

Some of his mathematical debts are visible in the architecture of his equations. For example, compare the infinite series of numbers that Archimedes used in his quadrature of the parabola,

$$\frac{4}{3} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots,$$

with the infinite series of symbols that Newton used in his quadrature of the hyperbola:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

If you plug x = −¼ into Newton's series, it becomes Archimedes's series. In that sense, Newton's series subsumes Archimedes's as a special case.

What's more, the similarity in their work extends to the geometric problems they considered. Both of them were fond of segments; Archimedes used his number series to square (or find the area of) a parabolic segment, whereas Newton used his jacked-up power series,

$$A_{\text{circular}}(x) = x - \tfrac{1}{6}x^3 - \tfrac{1}{40}x^5 - \tfrac{1}{112}x^7 - \tfrac{5}{1152}x^9 - \cdots,$$

to square a circular segment, and he used a different power series,

$$A_{\text{hyperbolic}}(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \cdots,$$

to square a hyperbolic segment.

Actually, Newton's series were infinitely more powerful than Archimedes's in that they enabled him to find the areas of not just one but a whole continuous infinity of circular and hyperbolic segments. That's what his abstract symbol x did for him. It let him change his problems continuously and effortlessly. It enabled him to tune the shape of segments by sliding x to the left or right so that what appeared to be a single infinite series was in fact an infinite family of infinite series, one for each choice of x. That was the power of power series. They let Newton solve infinitely many problems in a single stroke.

But again, he couldn't have done any of this without standing on the shoulders of giants. He unified, synthesized, and generalized ideas from his great predecessors: He inherited the Infinity Principle from Archimedes. He learned his tangent lines from Fermat. His decimals came from India. His variables came from Arabic algebra. His representation of curves as equations in the *xy* plane came from his reading of Descartes. His freewheeling shenanigans with infinity, his spirit of experimentation, and his openness to guesswork and induction came from Wallis. He mashed all of this together to create something fresh, something we're still using today to solve calculus problems: the versatile method of power series.